

# Generating permutations with a given major index

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## Abstract

In [S. Effler, F. Ruskey, A CAT algorithm for listing permutations with a given number of inversions, *I.P.L.*, 86/2 (2003)] the authors give an algorithm, which appears to be CAT, for generating permutations with a given major index. In the present paper we give a new algorithm for generating a Gray code for subexcedant sequences. We show that this algorithm is CAT and derive it into a CAT generating algorithm for a Gray code for permutations with a given major index.

## 1 Introduction

We present the first guaranteed constant average time generating algorithm for permutations with a fixed index. First we give a co-lex order generating algorithm for bounded compositions. Changing its generating order and specializing it for particular classes of compositions we derive a generating algorithms for a Gray code for fixed weight subexcedant sequences; and after some improvements we obtain an efficient version of this last algorithm. The generated Gray code has the remarkable property that two consecutive sequences differ in at most three adjacent positions and by a bounded amount in these positions. Finally applying a bijection introduced in [7] between subexcedant sequences and permutations with a given index we derive the desired algorithm, where consecutive generated permutations differ by at most three transpositions.

Often, Gray code generating algorithms can be re-expressed simpler as algorithms with the same time complexity and generating the same class of objects, but in different (*e.g.* lexicographical) order. This is not the case in our construction: the *Grayness* of the generated subexcedant sequences is critical in the construction of the efficient algorithm generating permutations with a fixed index.

A *statistic* on the set  $\mathfrak{S}_n$  of length  $n$  permutations is an association of an element of  $\mathbb{N}$  to each permutation in  $\mathfrak{S}_n$ . For  $\pi \in \mathfrak{S}_n$  the *major index*,  $\text{MAJ}$ , is a statistic defined by (see, for example, [3, Section 10.6])

$$\text{MAJ } \pi = \sum_{\substack{1 \leq i < n \\ \pi_i > \pi_{i+1}}} i.$$

**Definition 1.** For two integers  $n$  and  $k$ , an  $n$ -composition of  $k$  is an  $n$ -sequence  $\mathbf{c} = c_1 c_2 \dots c_n$  of non-negative integers with  $\sum_{i=1}^n c_i = k$ . For an  $n$ -sequence  $\mathbf{b} = b_1 b_2 \dots b_n$ ,  $\mathbf{c}$  is said  *$\mathbf{b}$ -bounded* if  $0 \leq c_i \leq b_i$ , for all  $i$ ,  $1 \leq i \leq n$ .

In this context  $b_1 b_2 \dots b_n$  is called *bounding sequence* and we will consider only bounding sequences with either  $b_i > 0$  or  $b_i = b_{i-1} = \dots = b_1 = 0$  for all  $i$ ,  $1 \leq i \leq n$ . Clearly,  $b_i = 0$  is equivalent to fix  $c_i = 0$ . We denote by  $C(k, n)$  the set of all  $n$ -compositions of  $k$ , and by  $C^{\mathbf{b}}(k, n)$  the set of  $\mathbf{b}$ -bounded  $n$ -compositions of  $k$ ; and if  $b_i \geq k$  for all  $i$ , then  $C^{\mathbf{b}}(k, n) = C(k, n)$ .

**Definition 2.** A *subexcedant sequence*  $\mathbf{c} = c_1 c_2 \dots c_n$  is an  $n$ -sequence with  $0 \leq c_i \leq i - 1$ , for all  $i$ ; and  $\sum_{i=1}^n c_i$  is called the *weight* of  $\mathbf{c}$ .

We denote by  $S(k, n)$  the set of length  $n$  and weight  $k$  subexcedant sequences, and clearly  $S(k, n) = C^{\mathbf{b}}(k, n)$  with  $\mathbf{b} = 0 1 2 \dots (n - 1)$ .

## 2 Generating fixed weight subexcedant sequences

We give three generating algorithms, and the third one generates efficiently combinatorial objects in bijection with permutations having fixed index :

- **Gen\_Collex** generates the set  $C^{\mathbf{b}}(k, n)$  of bounded compositions in co-lex order (defined later).
- **Gen1\_Gray** which is obtained from **Gen\_Collex** by:
  - changing its generating order, and
  - restricting it to the bounding sequence  $\mathbf{b} = 0 1 \dots (n - 1)$ .

It produces a Gray code for the set  $S(k, n)$ , and it can be seen as the definition of this Gray code.

- **Gen2\_Gray** is a an efficient version of **Gen1\_Gray**.

Finally, in Section 4, regarding the subexcedant sequences in  $S(k, n)$  as McMahon permutation codes (defined in Section 3), a constant average time generating algorithm for a Gray code for the set of permutations of length  $n$  with the major index equals  $k$  is obtained.

### 2.1 Algorithm Gen\_Collex

This algorithm generates  $C^{\mathbf{b}}(k, n)$  in *co-lex order*, which is defined as:  $c_1 c_2 \dots c_n$  precedes  $d_1 d_2 \dots d_n$  in co-lex order if  $c_n c_{n-1} \dots c_1$  precedes  $d_n d_{n-1} \dots d_1$  in lexicographical order. Its worst case time complexity is  $O(k)$  per composition.

For a set of bounded compositions  $C^{\mathbf{b}}(k, n)$ , an *increasable position* (with respect to  $C^{\mathbf{b}}(k, n)$ ) in a sequence  $c_1 c_2 \dots c_n \notin C^{\mathbf{b}}(k, n)$  is an index  $i$  such that:

- $c_1 = c_2 = \dots c_{i-1} = 0$ , and
- there is a composition  $d_1 d_2 \dots d_n \in C^{\mathbf{b}}(k, n)$  with  $c_i < d_i$  and  $c_{i+1} = d_{i+1}$ ,  $c_{i+2} = d_{i+2}$ ,  $\dots$ ,  $c_n = d_n$ .

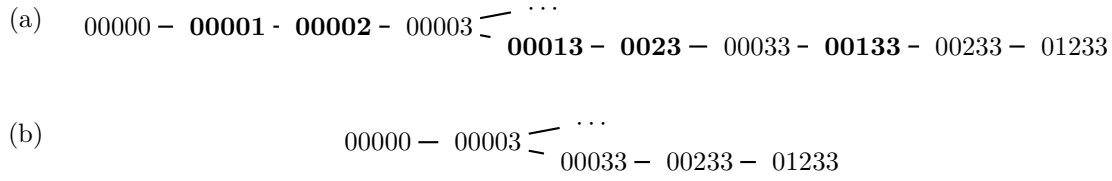


Figure 1: The path from the root 00000 to the composition 01233  $\in C^{01234}(9, 5)$ : (a) before deleting redundant nodes (in boldface); and (b) in the generating tree induced by the call of **Gen\_Colex**(9, 5) where redundant nodes are avoided.

For example, for  $C^{01233}(3, 5)$  the increasable positions are underlined in the following sequences: 00010 and 00200. Indeed, the first two positions in 00010 are not increasable since there is no composition in  $C^{01233}(3, 5)$  with the suffix 010; and the third position in 00200 is not increasable because 2 is the maximal value in this position. Clearly, if  $\ell < r$  are two increasable positions in  $\mathbf{c}$ , then each  $i$ ,  $\ell < i < r$ , is still an increasable position in  $\mathbf{c}$  (unless  $b_i = 0$ ).

Here is the sketch of the co-lex order generating procedure for  $C^b(k, n)$ :

- initialize  $\mathbf{c}$  by the length  $n$  sequence 00 ... 0;
- for each increasable position  $i$  in  $\mathbf{c}$ , increase  $c_i$  by one and call recursively the generating procedure if the obtained sequence  $\mathbf{c}$  is not a composition in  $C^b(k, n)$ , and output it elsewhere.

The complexity of the obtained algorithm is  $O(k)$  per generated composition and so inefficient. Indeed, too many nodes in the generating tree induced by this algorithm have degree one. Algorithm **Gen\_Colex** in Figure 2 avoids some of these nodes. We will identify a node in a generating tree by the corresponding value of the sequence  $\mathbf{c}$ ; and a *redundant node* in a generating tree induced by the previous sketched algorithm is a node with a unique successor and which differs in the same position from its ancestor and its successor. For example, in Figure 1 (a) redundant nodes are: 0001, 0002, 0013, 0023 and 0133. These nodes occur when, for a given suffix, the smallest value allowed in an increasable position in the current sequence  $\mathbf{c}$  is not 1, and this position is necessarily  $\ell$ , the leftmost increasable one. Algorithm **Gen\_Colex** avoids redundant nodes by setting  $c_\ell$  to its minimal value  $e = k - \sum_{j=1}^{\ell-1} b_j$  (and  $\sum_{j=1}^i b_j$  can be computed for each  $i$ ,  $1 \leq i \leq n$ , in a pre-processing step). For example, in Figure 1 (b) there are no redundant nodes. However, in the generating tree induced by **Gen\_Colex** there still remain arbitrary length sequences of successive nodes with a unique successor; they are avoided in procedure **Gen2\_Gray**.

Algorithm **Gen\_Colex** is given in Figure 2 where  $\ell$  is the leftmost increasable position in the current sequence  $\mathbf{c}$ , and  $r$  the leftmost non-zero position in  $\mathbf{c}$ , and thus the rightmost increasable position in  $\mathbf{c}$  is  $r$  if  $c_r < b_r$  and  $r - 1$  elsewhere ( $b_1 b_2 \dots b_n$  being the bounding sequence). The main call is **Gen\_Colex**( $k, n$ ) and initially  $\mathbf{c}$  is 00 ... 0. (As previously, in this algorithm the function  $k \mapsto \min\{s \mid \sum_{j=1}^s b_j \geq k\}$  can be computed and stored in an array, in a pre-processing step.)

The induced generating tree for the call **Gen\_Colex**(4, 5) is given in Figure 3 (a).

```

procedure Gen_Colex( $k, r$ )
  global  $n, c, b$ ;
  if  $k = 0$ 
  then print  $c$ ;
  else if  $c[r] = b[r]$ 
    then  $r := r - 1$ ;
  end if
   $\ell := \min\{s \mid \sum_{j=1}^s b[j] \geq k\}$ ;
  for  $i := \ell$  to  $r$  do
    if  $i = \ell$  then  $e := k - \sum_{j=1}^{\ell-1} b[j]$ ;
    else  $e := 1$ ;
    end if
     $c[i] := c[i] + e$ ;
    Gen_Colex( $k - e, i$ );
     $c[i] := c[i] - e$ ;
  end do
end if
end procedure.

```

Figure 2: Algorithm Gen\_Colex.

## 2.2 Algorithm Gen1\_Gray

This algorithm is defined in Figure 4 and is derived from **Gen\_Colex**: the order of recursive calls is changed according to a direction (parameter *dir*), and it is specialized for bounding sequences  $\mathbf{b} = 012 \dots (n-1)$ , and so it produces subexcedant sequences. It has the same time complexity as **Gen\_Colex** and we will show that it produces a Gray code.

The call of **Gen1\_Gray** with *dir* = 0 produces, in order, a recursive call with *dir* = 0, then  $r - \ell$  calls in the **for** statement with *dir* equals successively:

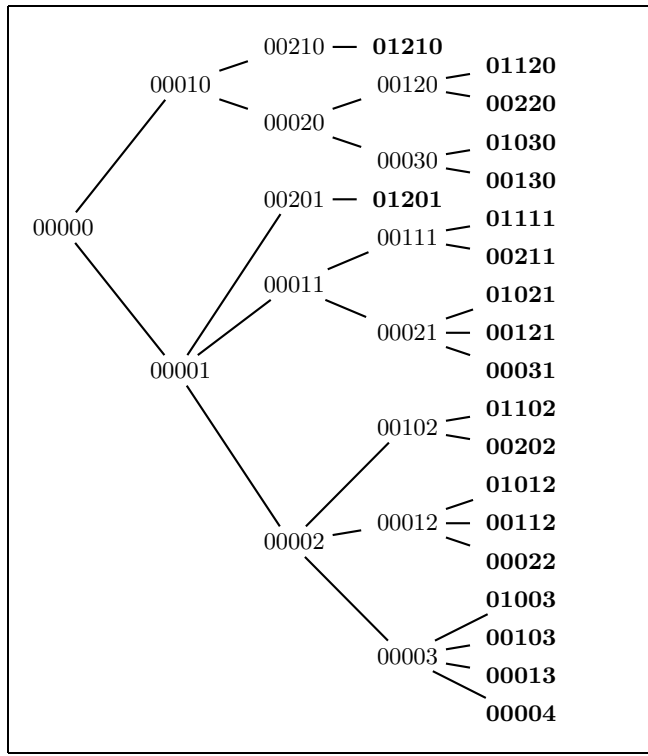
- 0, 1, ..., 0, 1, if  $r - \ell$  is even, and
- 1, 0, ..., 1, 0, 1, if  $r - \ell$  is odd.

In any case, the value of *dir* corresponding to the last call is 1.

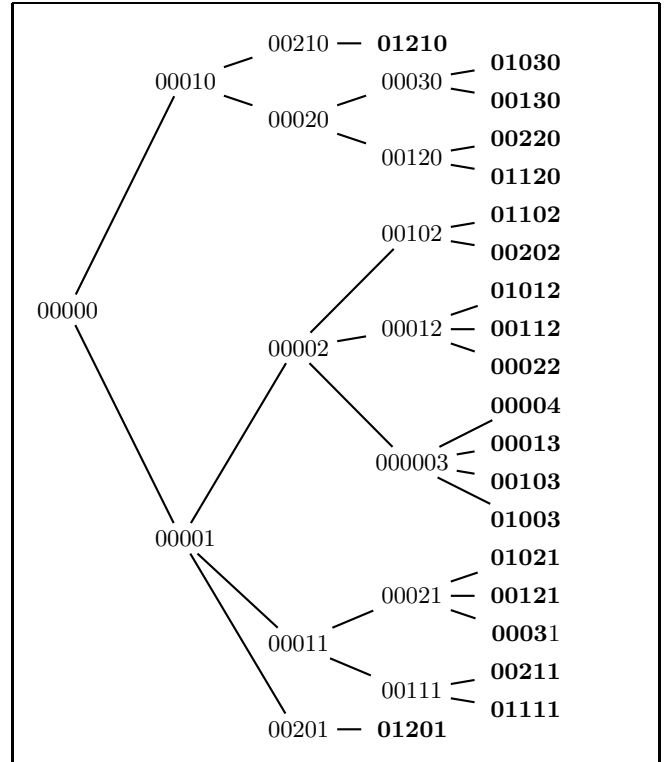
The call of **Gen1\_Gray** with *dir* = 1 produces the same operations as previously but in reverse order, and in each recursive call the value of *dir* is replaced by  $1 - \text{dir}$ . Thus, the call of **Gen1\_Gray** with *dir* = 1 produces, in order,  $r - \ell$  calls in the **for** statement with *dir* equals alternatively 0, 1, 0, ..., then a last call with *dir* = 1. See Figure 3 (b) for an example of generating tree induced by this procedure.

Let  $\mathcal{S}(k, n)$  be the *ordered list* for  $S(k, n)$  generated by the call **Gen1\_Gray**( $k, n, 0$ ), and it is easy to see that  $\mathcal{S}(k, n)$  is suffix partitioned, that is, sequences with the same suffix are contiguous; and Theorem 4 shows that  $\mathcal{S}(k, n)$  is a Gray code.

For a sequence  $\mathbf{c}$ , a  $k \geq 1$  and *dir*  $\in \{0, 1\}$  we denote by  $\text{first}(k; \text{dir}; \mathbf{c})$  and  $\text{last}(k; \text{dir}; \mathbf{c})$ , the first and last subexcedant sequence produced by the call of **Gen1\_Gray**( $k, r, \text{dir}$ ) if the current sequence is  $\mathbf{c}$ , and  $r$  the position of the leftmost non-zero value in  $\mathbf{c}$ . In particular, if  $\mathbf{c} = 00 \dots 0$ , then  $\text{first}(k; 0; \mathbf{c})$  is the first sequence in  $\mathcal{S}(k, n)$ , and  $\text{last}(k; 0; \mathbf{c})$  the last one.



(a)



(b)

Figure 3: (a): The tree induced by the call of `Gen_Colex(4,5)` with  $\mathbf{b} = 0\,1\,2\,3\,4$ , and (b): that induced by `Gen1_Gray(4,5)`. Terminal nodes are in bold-face

**Remark 1.**

1. For a sequence  $\mathbf{c}$ , the list produced by the call  $\text{Gen1\_Gray}(k, r, 0)$  is the reverse of the list produced by the call  $\text{Gen1\_Gray}(k, r, 1)$ , and with the previous notations we have

$$\text{last}(k; \text{dir}; \mathbf{c}) = \text{first}(k; 1 - \text{dir}; \mathbf{c}),$$

for  $\text{dir} \in \{0, 1\}$ .

2. Since the bounding sequence is  $\mathbf{b} = 0 \ 1 \ \dots \ (n-1)$  it follows that, for  $\mathbf{c} = 0 \ 0 \ \dots \ 0 \ c_i c_{i+1} \dots c_n$ ,  $c_i \neq 0$ ,  $\text{first}(k; 0; \mathbf{c})$  is

- $a_1 a_2 \dots a_{i-1} c_i c_{i+1} \dots c_n$  if  $k \leq \sum_{j=1}^{i-1} (j-1) = \frac{(i-1) \cdot (i-2)}{2}$ , where  $a_1 a_2 \dots a_{i-1}$  is the smallest sequence, in co-lex order, in  $S(k, i-1)$ ,
- $a_1 a_2 \dots a_i c_{i+1} \dots c_n$  if  $k > \frac{(i-1) \cdot (i-2)}{2}$ , where  $a_1 a_2 \dots a_i$  is the smallest sequence, in co-lex order, in  $S(k + c_i, i)$ .

```

procedure Gen1_Gray( $k, r, \text{dir}$ )
global  $n, c, b$ ;
if  $k = 0$ 
then output  $c$ ;
else if  $c[r] = r - 1$ 
then  $r := r - 1$ ;
end if
 $\ell := \min\{s \mid \frac{s(s-1)}{2} \geq k\}$ ;
 $e := k - \frac{(\ell-1)(\ell-2)}{2}$ ;
if  $\text{dir} = 0$ 
then  $c[\ell] := c[\ell] + e$ ; Gen1_Gray( $k - e, \ell, 0$ );  $c[\ell] := c[\ell] - e$ ;
 $\text{dir} := (r - \ell) \bmod 2$ ;
for  $i := \ell + 1$  to  $r$  do
 $c[i] := c[i] + 1$ ; Gen1_Gray( $k - 1, i, \text{dir}$ );  $\text{dir} := (\text{dir} + 1) \bmod 2$ ;  $c[i] := c[i] - 1$ ;
end do
else  $\text{dir} := 0$ ;
for  $i := r$  downto  $\ell + 1$  do
 $c[i] := c[i] + 1$ ; Gen1_Gray( $k - 1, i, \text{dir}$ );  $\text{dir} := (\text{dir} + 1) \bmod 2$ ;  $c[i] := c[i] - 1$ ;
end do
 $c[\ell] := c[\ell] + e$ ; Gen1_Gray( $k - e, \ell, 1$ );  $c[\ell] := c[\ell] - e$ ;
end if
end if
end procedure.

```

Figure 4: Algorithm **Gen1\_Gray**, the Gray code counterpart of **Gen\_Collex** specialized to subexcedant sequences.

Now we introduce the notion of close sequences. Roughly speaking, two sequences are close if they differ in at most three adjacent positions and by a bounded amount in these positions. Definition 3 below defines formally this notion, and Theorem 4 shows that consecutive subexcedant sequences generated by **Gen1\_Gray** are close.

Let  $\mathbf{s} = s_1 s_2 \dots s_n$  and  $\mathbf{t} = t_1 t_2 \dots t_n$  be two subexcedant sequences of same weight which differ in at most three adjacent positions, and let  $p$  be the rightmost of them (notice

that necessarily  $p \geq 3$ ). The *difference* between  $\mathbf{s}$  and  $\mathbf{t}$  is the 3-tuple

$$(a_1, a_2, a_3) = (s_{p-2} - t_{p-2}, s_{p-1} - t_{p-1}, s_p - t_p).$$

Since  $\mathbf{s}$  and  $\mathbf{t}$  have same weight it follows that  $a_1 + a_2 + a_3 = 0$ ; and we denote by  $-(a_1, a_2, a_3)$  the tuple  $(-a_1, -a_2, -a_3)$ .

**Definition 3.** Two sequences  $\mathbf{s}$  and  $\mathbf{t}$  in  $S(k, n)$  are *close* if:

- $\mathbf{s}$  and  $\mathbf{t}$  differ in at most three adjacent positions, and
- if  $(a_1, a_2, a_3)$  is the difference between  $\mathbf{s}$  and  $\mathbf{t}$ , then

$$(a_1, a_2, a_3) \in \{\pm(0, 1, -1), \pm(0, 2, -2), \pm(1, -2, 1), \pm(1, -3, 2), \pm(1, 1, -2), \pm(1, 0, -1)\}.$$

Even if the second point of this definition sound somewhat arbitrary, it turns out that consecutive sequences generated by algorithm **Gen1\_Gray** are close under this definition, and our generating algorithm for permutations with a given index in Section 4 is based on it.

**Example 1.** The following sequences are close: 01201 and 00301; 01003 and 01021; 00201 and 01011; 01132 and 01204; the positions where the sequences differ are underlined. Whereas the following sequences are not close: 00211 and 01030 (they differ in more than three positions); 01201 and 01030 (the difference 3-tuple is not a specified one).

**Remark 2.** If  $\mathbf{s}$  and  $\mathbf{t}$  are two close subexcedant sequences in  $S(k, n)$ , then there are at most two ‘intermediate’ subexcedant sequences  $\mathbf{s}'$ ,  $\mathbf{s}''$  in  $S(k, n)$  such that the differences between  $\mathbf{s}$  and  $\mathbf{s}'$ , between  $\mathbf{s}'$  and  $\mathbf{s}''$ , and  $\mathbf{s}''$  and  $\mathbf{t}$  are  $\pm(1, -1, 0)$ .

**Example 2.** Let  $\mathbf{s} = 010111$  and  $\mathbf{t} = 002011$  be two sequences in  $S(4, 6)$ . Then  $\mathbf{s}$  and  $\mathbf{t}$  are close since they difference is  $(1, -2, 1)$ , and there is one ‘intermediate’ sequence  $\mathbf{s}' = 001111$  in  $S(4, 6)$  with

- the difference between  $\mathbf{s}$  and  $\mathbf{s}'$  is  $(1, -1, 0)$ ,
- the difference between  $\mathbf{s}'$  and  $\mathbf{t}$  is  $(-1, 1, 0)$ .

A consequence of Remark 1.2 is:

**Remark 3.** If  $\mathbf{s}$  and  $\mathbf{t}$  are close subexcedant sequences and  $m$  is an integer such that both  $\mathbf{u} = \text{first}(m; 0; \mathbf{s})$  and  $\mathbf{v} = \text{first}(m; 0; \mathbf{t})$  exist, then  $\mathbf{u}$  and  $\mathbf{v}$  are also close.

**Theorem 4.** Two consecutive sequences in  $S(k, n)$  generated by the algorithm **Gen1\_Gray** are close.

*Proof.* Let  $\mathbf{s}$  and  $\mathbf{t}$  be two consecutive sequences generated by the call of **Gen1\_Gray**( $k, n, 0$ ). Then there is a sequence  $\mathbf{c} = c_1 c_2 \dots c_n$  and a recursive call of **Gen1\_Gray** acting on  $\mathbf{c}$  (referred later as the *root call* for  $\mathbf{s}$  and  $\mathbf{t}$ ) which produces, in the **for** statement, two calls so that  $\mathbf{s}$  is the last sequence produced by the first of them and  $\mathbf{t}$  the first produced by the second of them.

By Remark 1.1 it is enough to prove that  $\mathbf{s}$  and  $\mathbf{t}$  are close when their root call has direction 0.

Let  $\ell$  and  $r$ ,  $\ell \neq r$ , be the leftmost and the rightmost increasable positions in  $\mathbf{c}$  (and so  $c_1 = c_2 = \dots = c_{r-1} = 0$ , and possibly  $c_r = 0$ ); and  $i$  and  $i + 1$  be the positions where  $\mathbf{c}$  is modified by the root call in order to produce eventually  $\mathbf{s}$  and  $\mathbf{t}$ . Also we denote  $m = k - \sum_{j=1}^n c_j$  and  $e = m - \frac{\ell(\ell-1)}{2}$ .

We will give the shape of  $\mathbf{s}$  and  $\mathbf{t}$  according to the following four cases.

1.  $i = \ell$  and  $r - \ell$  is even,
2.  $i = \ell$  and  $r - \ell$  is odd,
3.  $i \neq \ell$  and the call corresponding to  $i$  in the **for** statement of the root call has direction 0 (and so that corresponding to  $i + 1$  has direction 1),
4.  $i \neq \ell$  and the call corresponding to  $i$  in the **for** statement of the root call has direction 1 (and so that corresponding to  $i + 1$  has direction 0).

Case 1.

$$\begin{aligned} \mathbf{s} &= \text{last}(m - e; 0; 00 \dots e c_{\ell+1} \dots c_n) \\ &= \text{first}(m - e; 1; 00 \dots e c_{\ell+1} \dots c_n) \\ &= \begin{cases} \text{first}(m - e - (\ell - 2); 0; 00 \dots (\ell - 2) e c_{\ell+1} \dots c_n) & \text{if } e = \ell - 1 \\ \text{first}(m - e - (\ell - 2); 0; 00 \dots (\ell - 3)(e + 1) c_{\ell+1} \dots c_n) & \text{if } e < \ell - 1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbf{t} &= \text{first}(m - 1; 0; 00 \dots (c_{\ell+1} + 1) \dots c_n) \\ &= \text{first}(m - e; 0; 00 \dots (e - 1)(c_{\ell+1} + 1) \dots c_n) \\ &= \text{first}(m - e - (\ell - 2); 0; 00 \dots (\ell - 2)(e - 1)(c_{\ell+1} + 1) \dots c_n). \end{aligned}$$

Case 2. In this case  $\mathbf{s}$  is the same as in the previous case and

$$\begin{aligned} \mathbf{t} &= \text{first}(m - 1; 1; 00 \dots 0(c_{\ell+1} + 1) \dots c_n) \\ &= \begin{cases} \text{first}(m - 2; 0; 00 \dots 0(c_{\ell+1} + 2) \dots c_n) & \text{if } c_{\ell+1} + 2 \leq \ell \\ \text{first}(m - e; 0; 00 \dots 0(e - 1)(c_{\ell+1} + 1) \dots c_n) & \text{if } c_{\ell+1} + 2 > \ell \end{cases} \\ &= \begin{cases} \text{first}(m - e - (\ell - 2); 0; 00 \dots 0(\ell - 2)(e - 2)(c_{\ell+1} + 2) \dots c_n) & \text{if } c_{\ell+1} + 2 \leq \ell \\ \text{first}(m - e - (\ell - 2); 0; 00 \dots (\ell - 2)(e - 1)(c_{\ell+1} + 1) \dots c_n) & \text{if } c_{\ell+1} + 2 > \ell. \end{cases} \end{aligned}$$

Case 3. In this case  $c_i = 0$  and

$$\begin{aligned} \mathbf{s} &= \text{last}(m - 1; 0; 00 \dots 01c_{i+1} \dots c_n) \\ &= \text{last}(m - 2; 1; 00 \dots 02c_{i+1} \dots c_n) \\ &= \text{first}(m - 2; 0; 00 \dots 02c_{i+1} \dots c_n), \end{aligned}$$

and

$$\begin{aligned} \mathbf{t} &= \text{first}(m - 1; 1; 00 \dots 0(c_{i+1} + 1) \dots c_n) \\ &= \begin{cases} \text{first}(m - 2; 0; 00 \dots 0(c_{i+1} + 2) \dots c_n) & \text{if } c_{i+1} + 2 \leq i \\ \text{first}(m - 2; 0; 00 \dots 1(c_{i+1} + 1) \dots c_n) & \text{if } c_{i+1} + 2 > i. \end{cases} \end{aligned}$$



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Figure 5: Four successive q-terminal nodes in the generating tree induced by the call **Gen1\_Gray**(11,7,0) which generates the list  $\mathcal{S}(11,7)$ .

Case 4. As previously,  $c_i = 0$  and

$$\begin{aligned} \mathbf{s} &= \text{last}(m-1; 1; 00 \dots 01c_{i+1} \dots c_n) \\ &= \text{first}(m-1; 0; 00 \dots 01c_{i+1} \dots c_n), \end{aligned}$$

and

$$\mathbf{t} = \text{first}(m-1; 0; 00 \dots 00(c_{i+1} + 1) \dots c_n).$$

Finally, by Remark 3 it follows that in each of the four cases  $\mathbf{s}$  and  $\mathbf{t}$  are close, and the statement holds.  $\square$

As a byproduct of the previous theorem and Remark 1.2 we have

**Remark 4.** If  $\mathbf{s} = s_1s_2 \dots s_n$  and  $\mathbf{t} = t_1t_2 \dots t_n$  are two consecutive sequences generated by **Gen1\_Gray** and  $p$  is the rightmost position where they differ, then  $s_1s_2 \dots s_{p-2}$  and  $t_1t_2 \dots t_{p-2}$  are the smallest, in co-lex order, sequences in  $S(x, p-2)$  and  $S(y, p-2)$ , respectively, with  $x = s_1 + s_2 + \dots + s_{p-2}$  and  $y = t_1 + t_2 + \dots + t_{p-2}$ . Remark that  $s_1s_2 \dots s_{p-2} = t_1t_2 \dots t_{p-2}$ , and so  $x = y$ , if  $\mathbf{s}$  and  $\mathbf{t}$  differ in two (adjacent) positions.

### 2.3 Algorithm Gen2\_Gray

Since the generating tree induced by the call of **Gen1\_Gray** contains still arbitrary length branches of nodes of degree one it has a poor time complexity. Here we show how some of these nodes can be avoided in order to obtain the efficient generating algorithm **Gen2\_Colex** presented in Figure 6.

A *quasi-terminal node* (*q-terminal node* for short) in the tree induced by a generating algorithm is defined recursively as: a q-terminal node is either a terminal node (node with no successor) or a node with only one successor which in turn is a q-terminal node. The q-terminal nodes occur for the calls of **Gen1\_Gray**( $k, r, dir$ ) when  $k = \frac{r(r-1)}{2}$ . See Figure 5 for an example.

The key improvement made by **Gen2\_Gray** consists in its last parameter  $p$ , which gives the rightmost position where the current sequence differ from its previous one in the list  $\mathcal{S}(k, n)$ , and **Gen2\_Gray** stops the recursive calls of more than three successive q-terminal calls. Thus, **Gen2\_Gray** generates only suffixes of the form  $c_{p-2}c_{p-1}c_p \dots c_n$ ; see Table 1 for an example. Since two consecutive sequences in the Gray code  $\mathcal{S}(k, n)$  differ in at most three adjacent positions, these suffixes are enough to generate efficiently  $\mathcal{S}(k, n)$ , and to generate (in Section 4) a Gray code for the set of length  $n$  permutations having the major index equal to  $k$ .

Now we explain how the parameter  $p$  propagates through recursive calls. A non terminal call of **Gen2\_Gray** produces one or several calls. The first of them (corresponding

```

procedure Gen2_Gray( $k, r, dir, p, u$ )
global  $n, c, b$ ;
if  $k = 0$  or  $(p - r) \geq 3$  and  $k = \frac{r(r-1)}{2}$ 
then output( $p, u$ );
else if  $c[r] = r - 1$ 
    then  $r := r - 1$ ;
    end if
     $\ell := \min\{s \mid \frac{s(s-1)}{2} \geq k\}$ ;
     $e := k - \frac{(\ell-1)(\ell-2)}{2}$ ;
    if  $dir = 0$ 
    then  $c[\ell] := c[\ell] + e$ ; Gen2_Gray( $k - e, \ell, 0, p, u$ );  $c[\ell] := c[\ell] - e$ ;
         $dir := (r - \ell) \bmod 2$ ;
        for  $i := \ell + 1$  to  $r$  do
             $c[i] := c[i] + 1$ ; Gen2_Gray( $k - 1, i, dir, i, k - 1 + c[i]$ );  $dir := (dir + 1) \bmod 2$ ;  $c[i] := c[i] - 1$ ;
        end do
    else  $dir := 0$ ;
        for  $i := r$  downto  $\ell + 1$  do
            if  $i = r$  then  $q := p$ ;  $v := u$ ; else  $q := i + 1$ ;  $v := c[i + 1] + k$ ; end if
             $c[i] := c[i] + 1$ ; Gen2_Gray( $k - 1, i, dir, q, v$ );  $dir := (dir + 1) \bmod 2$ ;  $c[i] := c[i] - 1$ ;
        end do
        if  $\ell = r$  then  $q := p$ ;  $v := u$ ; else  $q := \ell + 1$ ;  $v := c[\ell + 1] + k$ ; end if
         $c[\ell] := c[\ell] + e$ ; Gen2_Gray( $k - e, \ell, 1, q, v$ );  $c[\ell] := c[\ell] - e$ ;
    end if
end if
end procedure.

```

Figure 6: Algorithm Gen2\_Gray.

to a left child in the generating tree) inherits the value of the parameter  $p$  from its parent call; in the other calls the value of this parameter is the rightmost position where the current sequence differs from its previous generated one; this value is  $i$  if  $dir = 0$  and  $i + 1$  if  $dir = 1$ . So, each call keeps in the last parameter  $p$  the rightmost position where the current generated sequence differs from its previous one in the list  $\mathcal{S}(k, n)$ . Procedure Gen2\_Gray prevents to produce more than three successive q-terminal calls. For convenience, initially  $p = 0$ .

The last two parameters  $p$  and  $u$  of procedure Gen2\_Gray and output by it are used by procedure Update\_Perm in Section 4 in order to generate permutations with a given major index;  $u$  keeps the value of  $c_1 + c_2 + \dots + c_p$ , and for convenience, initially  $u = 0$ .

Even we will not make use later we sketch below an algorithm for efficiently generating the list  $\mathcal{S}(k, n)$ :

- initialize  $\mathbf{d}$  by the first sequence in  $\mathcal{S}(k, n)$ , i.e, the the smallest sequence in  $\mathcal{S}(k, n)$  in co-lex order, or equivalently, the largest one in lexicographical orders, and  $\mathbf{c}$  by  $00 \dots 0$ ,
- run Gen2\_Gray( $k, n, 0, 0, 0$ ) and for each  $p$  output by it update  $\mathbf{d}$  as:  $d[p-2] := c[p-2]$ ,  $d[p-1] := c[p-1]$ ,  $d[p] := c[p]$ .

#### Analyze of Gen2\_Gray

For a call of Gen2\_Gray( $k, r, dir, p, u$ ) necessarily  $k \leq \frac{r(r-1)}{2}$ , and if  $k > 0$  and

sequence	$p$	permutation	sequence	$p$	permutation
012100		214356	01 <u>00</u> 12	5	536124
0 <u>10</u> 300	4	324156	<u>00</u> 1012	3	635124
<u>00</u> 1300	3	423156	<u>000</u> 112	4	135624
002 <u>200</u>	4	413256	000 <u>022</u>	5	235614
<u>011</u> 200	3	314256	0000 <u>04</u>	6	345612
012 <u>010</u>	5	215346	0000 <u>13</u>	6	245613
<u>0111</u> 10	4	315246	000 <u>103</u>	5	145623
<u>002</u> 110	3	513246	<u>00</u> 1003	4	645123
000310	4	123546	<u>0100</u> 03	3	546123
<u>0012</u> 10	4	523146	010 <u>021</u>	6	436125
<u>0102</u> 10	3	325146	<u>0010</u> 21	3	634125
010030	5	435126	<u>0001</u> 21	4	134625
<u>0010</u> 30	3	534126	0000 <u>31</u>	5	234615
000 <u>130</u>	4	134526	000 <u>211</u>	5	124635
000040	5	234516	<u>0011</u> 11	4	624135
000220	5	124536	<u>0101</u> 11	3	426135
<u>0011</u> 20	4	524136	<u>0020</u> 11	4	614235
<u>0101</u> 20	3	425136	<u>0110</u> 11	3	416235
002020	4	514236	01 <u>1101</u>	5	316245
<u>0110</u> 20	3	415236	<u>0021</u> 01	3	613245
0110 <u>02</u>	6	516234	000 <u>301</u>	4	123645
<u>002002</u>	3	615234	00 <u>12</u> 01	4	623145
000202	4	125634	<u>0102</u> 01	3	326145
<u>0011</u> 02	4	625134	0 <u>12001</u>	4	216345
<u>0101</u> 02	3	526134			

Table 1: The subexcedant sequences generated by the call of `Gen1_Gray(4, 6, 0)` and their corresponding length 6 permutations with major index equals 4, permutations descent set is either  $\{1, 3\}$  or  $\{4\}$ . The three leftmost entries  $(c_{p-2}, c_{p-1}, c_p)$  updated by the call of `Gen2_Gray(4, 6, 0, 0, 0)` are underlined, where  $p$  is the rightmost position where a subexcedant sequence differ from its predecessor.

- $k \leq \frac{(r-1)(r-2)}{2}$ , then this call produces at least two recursive calls,
- $\frac{(r-1)(r-2)}{2} < k < \frac{r(r-1)}{2}$ , then this call produces a unique recursive call (of the form  $\text{Gen2\_Gray}(k', r, \cdot, \cdot, \cdot)$ , with  $k' = k - \frac{(r-1)(r-2)}{2}$ ), which in turn produce two calls,
- $k = \frac{r(r-1)}{2}$ , then this call is q-terminal call.

Sine the procedure **Gen2\_Gray** stops after three successive q-terminal calls, with a slight modification of Ruskey and van Baronaigien's [4] 'CAT' principle (see also [5]) it follows that **Gen2\_Gray** runs in constant amortized time.

### 3 The McMahon code of a permutation

Here we present the bijection  $\psi : S(n) \rightarrow \mathfrak{S}_n$ , introduced in [7], which have the following properties:

- the image through  $\psi$  of  $S(k, n)$  is the set of permutations in  $\mathfrak{S}_n$  with major index  $k$ ,
- $\psi$  is a 'Gray code preserving bijection' (see Theorem 6),
- $\tau$  is easily computed from  $\sigma$  and from the difference between  $\mathbf{s}$  and  $\mathbf{t}$ , the McMahon code of  $\sigma$  and  $\tau$ , if  $\mathbf{s}$  and  $\mathbf{t}$  are close.

In the next section we apply  $\psi$  in order to construct a list for the permutations in  $\mathfrak{S}_n$  with a major index equals  $k$  from the Gray code list  $\mathcal{S}(k, n)$ .

Let permutations act on indices, i.e., for  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$  and  $\tau = \tau_1 \tau_2 \dots \tau_n$  two permutations in  $\mathfrak{S}_n$ ,  $\sigma \cdot \tau = \sigma_{\tau_1} \sigma_{\tau_2} \dots \sigma_{\tau_n}$ . For a fixed integer  $n$ , let  $k$  and  $u$  be two integers,  $0 \leq k < u \leq n$ , and define  $[[u, k]] \in \mathfrak{S}_n$  as the permutation obtained after  $k$  right circular shifts of the length- $u$  prefix of the identity in  $\mathfrak{S}_n$ . In two line notation

$$[[u, k]] = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & u & u+1 & \dots & n \\ u-k+1 & u-k+2 & \dots & u & 1 & \dots & u-k & u+1 & \dots & n \end{pmatrix}.$$

For example, in  $\mathfrak{S}_5$  we have:  $[[3, 1]] = \underline{3} \underline{1} \underline{2} 4 5$ ,  $[[3, 2]] = \underline{2} \underline{3} \underline{1} 4 5$  and  $[[5, 3]] = \underline{3} \underline{4} \underline{5} \underline{1} \underline{2}$  (the rotated elements are underlined).

Let  $\psi : S(n) \rightarrow \mathfrak{S}_n$  be the function defined by

$$\begin{aligned} \psi(t_1 t_2 \dots t_n) &= [[n, t_n]] \cdot [[n-1, t_{n-1}]] \cdot \dots \cdot [[i, t_i]] \cdot \dots \cdot [[2, t_2]] \cdot [[1, t_1]] \\ &= \prod_{i=1}^n [[i, t_i]]. \end{aligned} \tag{1}$$

**Lemma 5** ([7]).

1. The function  $\psi$  defined above is a bijection.
2. For every  $\mathbf{t} = t_1 t_2 \dots t_n \in S(n)$ , we have  $\text{MAJ} \prod_{i=1}^n [[i, t_i]] = \sum_{i=1}^n t_i$ .

The first point of the previous lemma says that every permutation  $\pi \in \mathfrak{S}_n$  can be uniquely written as  $\prod_{i=n}^1 [[i, t_i]]$  for some  $t_i$ 's, and the subexcedant sequence  $t_1 t_2 \dots t_n$  is called the *McMahon code* of  $\pi$ . As a consequence of the second point of this lemma we have:

**Remark 5.** The restriction of  $\psi$  maps bijectively permutations in  $S(k, n)$  into permutations in  $\mathfrak{S}_n$  with major index equals  $k$ .

**Example 3.** The permutation  $\pi = 5\,2\,1\,6\,4\,3 \in \mathfrak{S}_n$  can be obtained from the identity by the following prefix rotations:

$$1\,2\,3\,4\,5\,6 \xrightarrow{[[6,3]]} 4\,5\,6\,1\,2\,3 \xrightarrow{[[5,4]]} 5\,6\,1\,2\,4\,3 \xrightarrow{[[4,2]]} 1\,2\,5\,6\,4\,3 \xrightarrow{[[3,2]]} 2\,5\,1\,6\,4\,3 \xrightarrow{[[2,1]]} 5\,2\,1\,6\,4\,3 \xrightarrow{[[1,0]]} 5\,2\,1\,6\,4\,3,$$

so

$$\pi = [[6, 3]] \cdot [[5, 4]] \cdot [[4, 2]] \cdot [[3, 2]] \cdot [[2, 1]] \cdot [[1, 0]],$$

and thus

$$\text{MAJ } \pi = 3 + 4 + 2 + 2 + 1 + 0 = 12.$$

Theorem 6 below states that if two permutations have their McMahon code differing in two adjacent positions, and by 1 and  $-1$  in these positions, then these permutations differ by the transposition of two entries. Before proving this theorem we need the following two propositions, where the transposition  $\langle u, v \rangle$  denote the permutation  $\pi$  (of convenient length) with  $\pi(i) = i$  for all  $i$ , except  $\pi(u) = v$  and  $\pi(v) = u$ .

**Proposition 1.** Let  $n, u$  and  $v$  be three integers,  $n \geq 3$ ,  $0 \leq u \leq n - 2$ ,  $1 \leq v \leq n - 2$ , and  $\sigma, \tau \in \mathfrak{S}_n$  defined by:

- $\sigma = [[n, u]] \cdot [[n - 1, v]]$ , and
- $\tau = [[n, u + 1]] \cdot [[n - 1, v - 1]]$ .

Then

$$\tau = \sigma \cdot \langle n, v \rangle.$$

*Proof.* First, remark that:

- $[[n, u + 1]]$ , is a right circular shift of  $[[n, u]]$ , and
- $[[n - 1, v - 1]]$  is a left circular shift of the first  $(n - 1)$  entries of  $[[n - 1, v]]$ ,

and so  $\sigma(i) = \tau(i)$  for all  $i$ ,  $1 \leq i \leq n$ , except for  $i = n$  and  $i = v$ . □

**Example 4.** For  $n = 7$ ,  $u = 4$  and  $v = 3$  we have

- $\sigma = [[n, u]] \cdot [[n - 1, v]] = [[7, 4]] \cdot [[6, 3]] = 7\,1\,2\,4\,5\,6\,3$ ,
- $\tau = [[n, u + 1]] \cdot [[n - 1, v - 1]] = [[7, 5]] \cdot [[6, 2]] = 7\,1\,3\,4\,5\,6\,2$ ,
- $\langle n, v \rangle = \langle 7, 3 \rangle$ ,

and  $\tau = \sigma \cdot \langle n, v \rangle$ .

**Proposition 2.** If  $\pi \in \mathfrak{S}_n$  and  $\langle u, v \rangle$  is a transposition in  $\mathfrak{S}_n$ , then

$$\pi^{-1} \cdot \langle u, v \rangle \cdot \pi = \langle \pi^{-1}(u), \pi^{-1}(v) \rangle.$$

*Proof.* Indeed,  $(\pi^{-1} \cdot \langle u, v \rangle \cdot \pi)(i) = i$ , for all  $i$ , except for  $i = \pi^{-1}(u)$  and  $i = \pi^{-1}(v)$ .  $\square$

**Theorem 6.** Let  $\sigma$  and  $\tau$  be two permutations in  $\mathfrak{S}_n$ ,  $n \geq 3$ , and  $\mathbf{s} = s_1 s_2 \dots s_n$  and  $\mathbf{t} = t_1 t_2 \dots t_n$  their McMahon codes. If there is a  $f$ ,  $2 \leq f \leq n-1$  such that  $t_i = s_i$  for all  $i$ , except  $t_f = s_f - 1$  and  $t_{f+1} = s_{f+1} + 1$ , then  $\tau$  and  $\sigma$  differ by a transposition. More precisely,

$$\tau = \sigma \cdot \langle \alpha^{-1}(u), \alpha^{-1}(v) \rangle$$

where

$$\alpha = \prod_{i=f-1}^1 [[i, s_i]] = \prod_{i=f-1}^1 [[i, t_i]],$$

and  $u = f + 1$ ,  $v = s_f$ .

*Proof.*

- $\tau = \prod_{i=n}^1 [[i, t_i]]$ , and so  $\tau \cdot \alpha^{-1} = \prod_{i=n}^f [[i, t_i]]$ , and
- $\sigma = \prod_{i=n}^1 [[i, s_i]]$ , and  $\sigma \cdot \alpha^{-1} = \prod_{i=n}^f [[i, s_i]]$ .

But, by Proposition 1,

$$\prod_{i=n}^f [[i, t_i]] = \prod_{i=n}^f [[i, s_i]] \cdot \langle f + 1, s_f \rangle$$

or, equivalently

$$\tau \cdot \alpha^{-1} = \sigma \cdot \alpha^{-1} \cdot \langle f + 1, s_f \rangle,$$

and by Proposition 2, the results holds.  $\square$

The previous theorem says that  $\sigma$  and  $\tau$  ‘have a small difference’ provided that their McMahon code,  $\mathbf{s}$  and  $\mathbf{t}$ , do so. Actually, we need that  $\mathbf{s}$  and  $\mathbf{t}$  are consecutive sequences in the list  $\mathcal{S}(k, n)$  and they have a more particular shape (see Remark 4). In this context, permutations having minimal McMahon code play a particular role.

It is routine to check the following proposition (see Figure 7 for an example).

**Proposition 3.** Let  $n$  and  $k$  be two integers,  $0 < k \leq \frac{n(n-1)}{2}$ ;  $\mathbf{a} = a_1 a_2 \dots a_n$  be the smallest subexcedant sequence in co-lex order with  $\sum_{i=1}^n a_i = k$ , and  $\alpha = \alpha_{n,k} = \psi(\mathbf{a})$  be the permutation in  $\mathfrak{S}_n$  having its McMahon code  $\mathbf{a}$ . Let  $j = \max \{i : a_i \neq 0\}$ , that is,  $\mathbf{a}$  has the form

$$012 \dots (j-3)(j-2)a_j 00 \dots 0.$$

Then

$$\alpha(i) = \begin{cases} j - a_j - i & \text{if } 1 \leq i \leq j - (a_j + 1), \\ 2j - a_j - i & \text{if } j - (a_j + 1) < i \leq j, \\ i & \text{if } i > j. \end{cases} \quad (2)$$

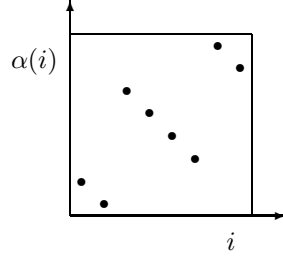


Figure 7: The permutation  $\alpha = 2\,1\,6\,5\,4\,3\,8\,7$  with the McMahon code  $\mathbf{a} = 0\,1\,2\,3\,4\,3\,0\,0$ , the the smallest, in co-lex order, subexcedant sequence in  $S(13, 8)$ , see Proposition 3.

**Remark 6.** The permutation  $\alpha$  defined in Proposition in 3 is an involution, that is  $\alpha^{-1} = \alpha$ .

Combining Proposition 3 and Remark 6, Theorem 6 becomes in particular

**Proposition 4.** *Let  $\sigma$ ,  $\tau$ ,  $\mathbf{s}$  and  $\mathbf{t}$  be as in Theorem 6. In addition, let suppose that there is a  $j$ ,  $0 \leq j \leq f-1$ , such that*

1.  $s_i = t_i = 0$  for  $j < i \leq f-1$ , and
2. if  $j > 0$ , then
  - $s_j = t_j \neq 0$ , and
  - $s_i = t_i = i-1$  for  $1 \leq i < j$ .

Then

$$\tau = \sigma \cdot \langle \phi_j(f+1), \phi_j(s_f) \rangle$$

with

$$\phi_j(i) = \begin{cases} j - s_j - i & \text{if } 1 \leq i \leq j - (s_j + 1), \\ 2j - s_j - i & \text{if } j - (s_j + 1) < i \leq j, \\ i & \text{if } i > j. \end{cases} \quad (3)$$

Notice that, the conditions 1 and 2 in the previous proposition require that  $s_1 s_2 \dots s_{f-1} = t_1 t_2 \dots t_{f-1}$  be the smallest subexcedant sequence, in co-lex order, in  $S(f-1)$  with fixed value for  $\sum_{i=1}^{f-1} s_i = \sum_{i=1}^{f-1} t_i$ . Also, for point 2, necessarily  $j \geq 2$ .

## 4 Generating permutations with a given major index

Let  $\sigma$  and  $\tau$  be two permutations with their McMahon code  $\mathbf{s} = s_1 s_2 \dots s_n$  and  $\mathbf{t} = t_1 t_2 \dots t_n$  belonging to  $S(k, n)$ , and differing in positions  $f$  and  $f+1$  by 1 and  $-1$  in these positions.

Let

- $v = s_f - t_f \in \{-1, 1\}$ , and

- $x = \sum_{i=1}^{f-1} s_i = \sum_{i=1}^{f-1} t_i$ .

If  $s_1 s_2 \dots s_{f-1}$  is the smallest sequence in  $S(x, f-1)$ , in co-lex order, then applying Proposition 4 it follows that the run of the procedure **transp**( $v, f, x$ ) defined in Figure 8 transforms  $\sigma$  into  $\tau$  and  $\mathbf{s}$  into  $\mathbf{t}$ .

```

procedure transp( $v, f, x$ )
   $j := \min\{i : \frac{i(i-1)}{2} \geq x\}$ ;
  if  $v = 1$ 
  then  $\sigma := \sigma \cdot \langle \phi_j(f+1), \phi_j(s[f]) \rangle$ ;
  else  $\sigma := \sigma \cdot \langle \phi_j(f+1), \phi_j(s[f]+1) \rangle$ ;
  endif
   $s[f] := s[f] - v$ ;
   $s[f+1] := s[f+1] + v$ ;
end procedure.

```

Figure 8: Algorithm **transp**, where  $\phi_j$  is defined in relation (3).

Let now  $f$  be the leftmost position where two consecutive sequences  $\mathbf{s}$  and  $\mathbf{t}$  in the list  $\mathcal{S}(k, n)$  differ, and  $\sigma$  and  $\tau$  be the permutations having they McMahon code  $\mathbf{s}$  and  $\mathbf{t}$ . By Remarks 2 and 4 we have that, repeated calls of **transp** transform  $\mathbf{s}$  into  $\mathbf{t}$ , and  $\sigma$  into  $\tau$ . This is true for each possible 3-tuples given in Definition 3 and corresponding to two consecutive subexcedant sequences in  $\mathcal{S}(k, n)$ , and algorithm **Update\_Perm** in Figure 9 exhausts all these 3-tuples.

For example, if  $\mathbf{s}$  and  $\mathbf{t}$  are the two sequences in Example 2 with they difference  $(1, -2, 1)$ ,  $f = 2$  and  $x = 0$ , then the calls

```

transp(1,  $f$ ,  $x$ );
transp(-1,  $f+1$ ,  $x + s[f]$ );

```

transform  $\mathbf{s}$  into  $\mathbf{t}$  and  $\sigma$  into  $\tau$ .

Algorithm **Gen2\_Gray** provides  $p$ , the rightmost position where the current sequence  $\mathbf{c}$  differs from the previous generated one, and  $u = \sum_{i=1}^p c_i$ . Algorithm **Update\_Perm** uses  $f$ , the leftmost position where  $\mathbf{c}$  differs from the previous generated sequence, and  $x = \sum_{i=1}^{f-1} c_i$ .

Now, we sketch the generating algorithm for the set of permutations in  $\mathfrak{S}_n$  having index  $k$ .

- initialize  $\mathbf{s}$  by the smallest, in co-lex order, sequence in  $S(k, n)$  and  $\sigma$  by the permutation in  $\mathfrak{S}_n$  having its McMahon code  $\mathbf{s}$ ,
- run **Gen2\_Gray**( $k, n, 0, 0, 0$ ) where **output**( $p, u$ ) is replaced by **Update\_Perm**( $p, u$ ).

The obtained list of permutations is the image of the Gray code  $\mathcal{S}(k, n)$  through the bijection  $\psi$  defined in relation (1); it consists of all permutations in  $\mathfrak{S}_n$  with major index equal to  $k$ , and two consecutive permutations differ by at most three transpositions. See Table 1 for the list of permutations in  $\mathfrak{S}_6$  and with major index 4.



```

procedure Update_Perm( $p, u$ )
   $x := u - c[p] - c[p - 1]$ ;
  if  $p - 2 \geq 1$  and  $s[p - 2] = c[p - 2]$ 
  then  $f := p - 1$ ;
  else  $f := p - 2$ ;  $x := x - c[f]$ ;
  endif
   $(a_1, a_2) := (s[f] - c[f], s[f + 1] - c[f + 1])$ ;
  if  $f + 2 > n$  then  $a_3 := 0$ ; else  $a_3 := s[f + 2] - c[f + 2]$ ; endif
  if  $a_1 > 0$  then  $v := 1$ ; else  $v := -1$ ; endif
  case  $(a_1, a_2, a_3)$  of
     $\pm(1, -1, 0)$  : transp( $v, f, x$ );
     $\pm(2, -2, 0)$  : transp( $v, f, x$ ); transp( $v, f, x$ )
     $\pm(1, -2, 1)$  : transp( $v, f, x$ ); transp( $-v, f + 1, x + s[f]$ );
     $\pm(1, -3, 2)$  : transp( $v, f, x$ ); transp( $-v, f + 1, x + s[f]$ ); transp( $-v, f + 1, x + s[f]$ );
     $\pm(1, 1, -2)$  : transp( $v, f + 1, x + s[f]$ ); transp( $v, f, x$ ); transp( $v, f + 1, x + s[f]$ );
     $(1, 0, -1)$  : transp( $1, f, x$ ); transp( $1, f + 1, x + s[f]$ );
     $(-1, 0, 1)$  : transp( $-1, f + 1, x + s[f]$ ); transp( $-1, f, x$ );
  end case
end procedure.

```

Figure 9: Algorithm Update\_Perm.

## 5 Final remarks

Numerical evidences show that if we change the generating order of algorithm **Gen\_Collex** as for **Gen1\_Gray**, but without restricting it to subexcedant sequences, then the obtained list for bounded compositions is still a Gray code with the closeness definition slightly relaxed: two consecutive compositions differ in at most four adjacent positions. Also, T. Walsh gives in [8] an efficient generating algorithm for a Gray code for bounded compositions of an integer, and in particular for subexcedant sequences. In this Gray code two consecutive sequences differ in two positions and by 1 and  $-1$  in these positions; but these positions can be arbitrarily far, and so the image of this Gray code through the bijection  $\psi$  defined by relation (1) in Section 3 does not give a Gray code for permutations with a fixed index.

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